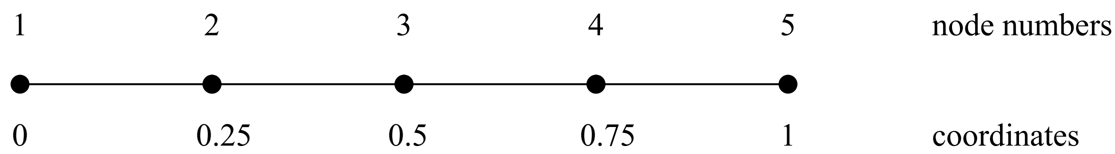
ME235A Finite Element Analysis
Fall, 2001

Solution to Problem Set #2
Oct. 17 2001

Exercise 1 on page 36 (The problem is not restated.)



h = 0.25 for all nodes

a. The shape functions are:

$$N_1 = \begin{cases} (x_2 - x)/h & x_1 \le x \le x_2 \\ 0 & elsewhere \end{cases}$$

$$N_A = \begin{cases} (x - x_{A-1})/h & x_{A-1} \le x \le x_A \\ (x_{A+1} - x)/h & x_A \le x \le x_{A+1} \\ 0 & elsewhere \end{cases} \qquad (2 § A § 4)$$

$$N_5 = \begin{cases} (x - x_4)/h & x_4 \le x \le x_5 \\ 0 & elsewhere \end{cases}$$

The stiffness matrix **K** is a 4 µ 4 symmetric matrix, whose elements are given as:

$$K_{AB} = a(N_A, N_B) = \int_0^1 N_{A,x} N_{B,x} dx$$

So that

$$K_{11} = \int_{x_1}^{x_2} (-1/h)^2 \, dx = 1/h = 4$$

$$K_{12} = \int_{x_1}^{x_2} (-1/h)(1/h) dx = -4$$

$$K_{22} = \int_{x_1}^{x_2} (1/h)^2 \, dx + \int_{x_2}^{x_3} (-1/h)^2 \, dx = 2/h = 8$$

etc.

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix}$$

The force vector **F** is 4 µ 1 with elements given as:

$$F_A = (N_A, f) + N_A(0)h - a(N_A, N_5)g = q(N_A, x)$$

So that,

$$F_1 = q/h \int_{x_1}^{x_2} (x_2 - x)x\,dx = q/96$$

$$F_2 = q/h \left(\int_{x_1}^{x_2} (x - x_1)x\,dx + \int_{x_2}^{x_3} (x_3 - x)x\,dx \right) = q/16$$

$$F_3 = q/h \left(\int_{x_2}^{x_3} (x - x_2)x\,dx + \int_{x_3}^{x_4} (x_4 - x)x\,dx \right) = q/8$$

$$F_4 = q/h \left(\int_{x_3}^{x_4} (x - x_3)x\,dx + \int_{x_4}^{x_5} (x_5 - x)x\,dx \right) = 3q/16$$

The force vector is

$$\mathbf{F} = q \begin{Bmatrix} 1/96 \\ 1/16 \\ 1/8 \\ 3/16 \end{Bmatrix}$$

Now, solving the matrix equation $\mathbf{Kd} = \mathbf{F}$ for **d**, we have

$$\mathbf{d} = q \begin{Bmatrix} 1/6 \\ 21/128 \\ 7/48 \\ 37/384 \end{Bmatrix} = q \begin{Bmatrix} 0.1667 \\ 0.1641 \\ 0.1458 \\ 0.0964 \end{Bmatrix}$$

The exact solution is

$$u(x) = q(1 - x^3)/6$$

Check that $u(x_A) = d_A$.

b. The derivative of u^h in each element $]x_A, x_{A+1}[$ is constant and given by:

$$u^h_{,x}(x) = \frac{u^h(x_{A+1}) - u^h(x_A)}{h} = \frac{u(x_{A+1}) - u(x_A)}{h}, \quad x \in]x_A, x_{A+1}[$$

$$u_{,x}(x) = -\frac{qx^2}{2}$$

element 1: $re_{,x} = \dfrac{|u^h_{,x}(1/8) - u_{,x}(1/8)|}{q/2} = \dfrac{1}{192}$

same for elements 2, 3 and 4

c. For $h = 1$ ($n = 1$),

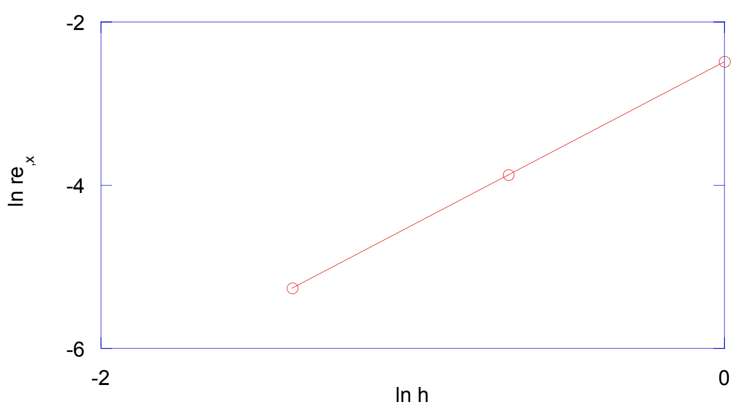
$$u^h_{,x}(x) = -\frac{q}{6}$$

$$re_{,x} = \frac{|u^h_{,x} - u_{,x}(1/2)|}{q/2} = \frac{1}{12}$$

For $h = 1/2$ ($n = 2$) (consider element 1 only),

$$u^h_{,x} = -\frac{q}{24}$$

$$re_{,x} = \frac{|u^h_{,x} - u_{,x}(1/4)|}{q/2} = \frac{1}{48}$$



d. $$re_{,x} = ch^k$$

$$\ln re_{,x} = k \ln h + \ln c$$

(i) The slope of the curve is k, and is the order of convergence (accuracy). This indicates the rate of convergence of the derivative as the mesh is refined.

(ii) The y intercept is log c, and indicates the relative error when h = 1, i.e., when the entire bar is treated as one element. This is the largest error in the derivative.