ME235A Finite Element Anlysis
Fall, 2001

Solution to Problem Set #3
Oct. 24 2001

Exercise 2 (page 46) (The problem is not restated.)

i. The variational equation is

$$a(w^h, u^h) + (w^h, \lambda u^h) = (w^h, f) + w^h(0)h$$

Let $u^h = v^h + g^h$, then,

$$a(w^h, v^h) + (w^h, \lambda v^h) = (w^h, f) + w^h(0)h - a(w^h, g^h) - (w, \lambda g^h)$$

ii. Let $w^h = \sum_{A=1}^{n} c_A N_A$ and $v^h = \sum_{A=1}^{n} d_A N_A$

$$a(\sum_{A=1}^{n} c_A N_A, \sum_{B=1}^{n} d_B N_B) + (\sum_{A=1}^{n} c_A N_A, \lambda \sum_{B=1}^{n} d_B N_B)$$

$$= (\sum_{A=1}^{n} c_A N_A, f) + \sum_{A=1}^{n} c_A N_A(0)h - a(\sum_{A=1}^{n} c_A N_A, g^h) - (\sum_{A=1}^{n} c_A N_A, \lambda g^h)$$

$$\sum_{A=1}^{n} c_A \left[\sum_{B=1}^{n} d_B \left(a(N_A, N_B) + (N_A, \lambda N_B) \right) \right] = \sum_{A=1}^{n} c_A \left[(N_A, f) + N_A(0)h - a(N_A, g^h) - (N_A, \lambda g^h) \right]$$

This can be written as $\sum_{B=1}^{n} K_{AB} d_B = F_A \qquad A = 1, \ldots, n$

where $K_{AB} = a(N_A, N_B) + (N_A, \lambda N_B)$.

The element stiffness matrix $\boldsymbol{k}^e = \left[k_{ab}^e \right]$ is given as

$$k_{ab}^e = a(N_a, N_b)^e + (N_a, \lambda N_2)^e$$

iii.

$$k_{ab}^e = \int_{-1}^{+1} N_{a,\xi}(\xi) N_{b,\xi}(\xi)(x_{,\xi}(\xi))^{-1} d\xi + \lambda \int_{-1}^{+1} N_a(\xi) N_b(\xi) x_{,\xi}(\xi) d\xi$$

$$\boldsymbol{k}^e = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\lambda h^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{h^e} + \frac{\lambda h^e}{3} & -\frac{1}{h^e} + \frac{\lambda h^e}{6} \\ -\frac{1}{h^e} + \frac{\lambda h^e}{6} & \frac{1}{h^e} + \frac{\lambda h^e}{3} \end{bmatrix}$$

iv.

$$K_{AB} = a(N_A, N_B) + (N_A, \lambda N_B) = a(N_A, N_B) + (N_A, \lambda N_B)$$
$$= a(N_B, N_A) + \lambda(N_B, N_A) = a(N_B, N_A) + (N_B, \lambda N_A) = K_{BA}$$

v. Let $\boldsymbol{c} = \{c_A\}$, $A = 1, \ldots, n$. Use the components of $\boldsymbol{c}$ to construct a vector w^h in V^h.

$$w^h = \sum_{A=1}^{n} c_A N_A$$

$$c^T K c = \sum_{A,B=1}^{n} c_A K_{AB} c_B = \sum_{A,B=1}^{n} c_A \left[a(N_A, N_B) + \lambda(N_A, N_N) \right] c_B$$

$$= a(\sum_{A=1}^{n} c_A N_A, \sum_{B=1}^{n} c_B N_B) + \lambda(\sum_{A=1}^{n} c_A N_A, \sum_{B=1}^{n} c_B N_B) = a(w^h, w^h) + \lambda(w^h, w^h)$$

$$= \int_0^1 (w^h_{,x})^2\, dx + \lambda \int_0^1 (w^h)^2\, dx \quad \geq 0$$

Assume $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0$. Then,

$$\int_0^1 (\underbrace{w^h_{,x}}_{0})^2\, dx + \lambda \int_0^1 (\underbrace{w^h}_{0})^2\, dx = 0 \qquad (*)$$

So, $w^h(x) = 0$, for all $x \in [0, 1]$, which means that $c_A = 0$, $A = 1, \ldots, n$. Therefore, $\boldsymbol{c} = \boldsymbol{0}$. It is unnecessary to use the boundary condition $w^h(1) = 0$, as since λ is positive, the second term of (*) leads to $w^h = 0$.

vi. $a(u - u^h, w^h) + \lambda(u - u^h, w^h) = 0 \qquad \forall w^h \in \mathcal{V}^h$

$$a(u - u^h, g) + \lambda(u - u^h, g) = (u - u^h, \delta_y) = u(y) - u^h(y)$$

However, since $g \notin \mathcal{V}^h$ for the linear basis functions,

$$u(x_A) - u^h(x_A) \neq 0$$

The solution is not nodally exact.

vii. $u^h(x) = c_1 e^{px} + c_2 e^{-px}$

which must satisfy the conditions,

$$u^h(x_1^e) = d_1^e = c_1 \exp(px_1^e) + c_2 \exp(-px_1^e)$$

$$u^h(x_2^e) = d_2^e = c_1 \exp(px_2^e) + c_2 \exp(-px_2^e)$$

write the above equations in matrix form,

$$\begin{bmatrix} \exp(px_1^e) & \exp(-px_1^e) \\ \exp(px_2^e) & \exp(-px_2^e) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} d_1^e \\ d_2^e \end{bmatrix}$$

Solve for the constant c's

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{d_1^e \exp(px_1^e) - d_2^e \exp(px_2^e)}{\exp(2px_1^e) - \exp(2px_2^e)} \\ \dfrac{-d_1^e \exp\left[p(x_1^e + 2x_2^e)\right] + d_2^e \exp\left[p(2x_1^e + x_2^e)\right]}{\exp(2px_1^e) - \exp(2px_2^e)} \end{bmatrix}$$

We have

$$u^h(x) = d_1^e N_1(x) + d_2^e N_2(x)$$

where

$$N_1(x) = \frac{\exp(px_1^e)\exp(px) - \exp\left[p(x_1^e + 2x_2^e)\right]\exp(-px)}{\exp(2px_1^e) - \exp(2px_2^e)}$$

$$N_2(x) = \frac{-\exp(px_2^e)\exp(px) + \exp\left[p(2x_1^e + x_2^e)\right]\exp(-px)}{\exp(2px_1^e) - \exp(2px_2^e)}$$

If we can show $g \in \mathcal{V}^h$ when y is a nodal point, the finite element solution is nodally exact. (See **vi**.)

$$g(x;y)=\begin{cases} c_1 e^{px}+c_2 e^{-px} & 0\le x\le y \\ c_3 e^{px}+c_4 e^{-px} & y\le x\le 1\end{cases}$$

where the c's are determined from the boundary conditions as

$$c_1=c_2=\frac{e^{-py}}{p-1}$$

$$c_3=\frac{e^{-py}(1+e^{-py})}{(p-1)(e^{2py}-e^{2p})}$$

$$c_4=-\frac{e^{-p(y-2)}(1+e^{2py})}{(p-1)(e^{2py}-e^{2p})}$$

Now we can show that $g(x)$ can be represented as a linear combination of the basis functions N_1 and N_2, *i.e.*,

$$g_1(x)=\frac{e^{-py}}{p-1}(e^{px}+e^{-px})=AN_1(x)+BN_2(x)$$

$$g_2(x)=\frac{e^{-py}(1+e^{-py})}{(p-1)(e^{2py}-e^{2p})}e^{px}-\frac{e^{-p(y-2)}(1+e^{2py})}{(p-1)(e^{2py}-e^{2p})}e^{-px}=CN_1(x)+DN_2(x)$$

where

$$A=\frac{e^{-py}(e^{3px_1}+e^{px_2})(e^{2px_1}-e^{2px_2})}{(p-1)(e^{4px_1}-e^{p(x_1+3x_2)})}$$

$$B=\frac{e^{-py}(e^{px_1}+e^{px_2})(1+e^{2px_2})}{(p-1)(e^{2px_1}+e^{2px_2}+e^{p(x_1+x_2)})}$$

$$C=\frac{e^{-2py}(e^{2px_1}-e^{2px_2})(e^{3px_1}+e^{p(3x_1+y)}+e^{p(2+x_2+y)}+e^{p(2+x_2+3y})}{(p-1)(e^{4px_1}-e^{p(x_1+3x_2)})(e^{2py}-e^{2p})}$$

$$D=\frac{e^{-2py}(e^{px_1}+e^{px_2})(e^{2px_2}+e^{p(2+y)}+e^{p(2x_2+y)}+e^{p(2+3y)})}{(p-1)(e^{2px_1}+e^{2px_2}+e^{p(x_1+x_2)})(e^{2py}-e^{2p})}$$

so that, if y is a nodal point, g œ V^h, and the finite element solution is exact at the nodes.

$$u(x_a^e)-u^h(x_a^e)=0$$

Section 1.16 (page 49) (The problem is not restated.)

a. i. Assume u œ H^2 is a solution of (S). Then, since $u(1)=0$ and $u_{,x}(1)=0$, u œ S. For any function w œ V,

$$\int_0^1 w\left(EIu_{,xxxx}-f\right)dx=0$$

Integrating by parts twice, and using the prescribed moment and shear boundary conditions and that $w(1)=0$, $w_{,x}(1)=0$,

$$0=-\int_0^1 w_{,x}EIu_{,xxx}dx-\int_0^1 wfdx-w(0)EIu_{,xxx}(0)$$

$$= \int_0^1 w_{,xx} EIu_{,xx} dx - \int_0^1 wfdx - w(0)EIu_{,xxx}(0) + w_{,x}(0)EIu_{,xx}(0)$$

$$= \int_0^1 w_{,xx} EIu_{,xx} dx - \int_0^1 wfdx - w(0)Q + w_{,x}(0)M$$

for all $w \in \mathcal{V}$. Therefore, u is a solution of (W).

ii. Assume u is a solution of (W). Since $u \in \mathcal{S}$, $u(1) = 0$ and $u_{,x}(1) = 0$.

$$\int_0^1 w_{,xx} EIu_{,xx} dx = \int_0^1 wfdx - w_{,x}(0)M + w(0)Q$$

Integrating by parts twice, and using the fact that $w(1) = 0$ and $w_{,x}(1) = 0$,

$$0 = -\int_0^1 w_{,x} EIu_{,xxx} dx - \int_0^1 wfdx + w_{,x}(0)M - w(0)Q - w_{,x}(0)EIu_{,xx}$$

$$= \int_0^1 w(EIu_{,xxxx} - f)dx + w_{,x}(0)(M - EIu_{,xx}(0)) - w(0)(Q - EIu_{,xxx}(0))$$

Take $w = \phi(EIu_{,xxxx} - f)$, where ϕ is smooth, $\phi(x) > 0$ for $x \in \Omega$, and $\phi(1) = \phi_{,x}(1) = \phi(0) = \phi_{,x}(0) = 0$. Then,

$$0 = \int_0^1 \phi \underbrace{(EIu_{,xxxx} - f)^2}_{\geq 0} dx$$

Since $\phi > 0$ on Ω,

$$EIu_{,xxxx} = f \qquad \text{on } \Omega$$

Now, since in general, $w(0) \neq 0$ and $w_{,x}(0) \neq 0$, we have

$$\left.\begin{aligned} EIu_{,xx}(0) &= M \\ EIu_{,xxx}(0) &= Q \end{aligned}\right\} \text{natural boundary conditions}$$

Therefore, u is a solution of (S).

b. $$w^h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

which must satisfy

$$w^h(x_1) = c_1 + c_2 x_1 + c_3 x_1^2 + c_4 x_1^3$$
$$w^h(x_2) = c_1 + c_2 x_2 + c_3 x_2^2 + c_4 x_2^3$$
$$w^h_{,x}(x_1) = c_2 + 2c_3 x_1 + 3c_4 x_1^2$$
$$w^h_{,x}(x_2) = c_2 + 2c_3 x_2 + 3c_4 x_2^2$$

These conditions can be written as

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} w^h(x_1) \\ w^h(x_2) \\ w^h_{,x}(x_1) \\ w^h_{,2}(x_2) \end{bmatrix}$$

Solving for the c's, we get

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -w^h(x_1)\dfrac{(3x_1-x_2)x_2^2}{h^3} - w^h(x_2)\dfrac{x_1^2(x_1-3x_2)}{h^3} - w^h_{,x}(x_1)\dfrac{x_1x_2^2}{h^2} - w^h_{,x}(x_2)\dfrac{x_1^2x_2}{h^2} \\ w^h(x_1)\dfrac{6x_1x_2}{h^3} - w^h(x_2)\dfrac{6x_1x_2}{h^3} + w^h_{,x}(x_1)\dfrac{x_2(2x_1+x_2)}{h^2} + w^h_{,x}(x_2)\dfrac{x_1(x_1+2x_2)}{h^2} \\ -w^h(x_1)\dfrac{3(x_1+x_2)}{h^3} + w^h(x_2)\dfrac{3(x_1+x_2)}{h^3} - w^h_{,x}(x_1)\dfrac{x_1+2x_2}{h^2} - w^h_{,x}(x_2)\dfrac{2x_1+x_2}{h^2} \\ w^h(x_1)\dfrac{2}{h^3} - w^h(x_2)\dfrac{2}{h^3} + w^h_{,x}(x_1)\dfrac{1}{h^2} + w^h_{,x}(x_2)\dfrac{1}{h^2} \end{bmatrix}$$

Now, we have

$$w^h(x) = N_1(x)w^h(x_1) + N_3(x)w^h(x_2) + N_2(x)w^h_{,x}(x_1) + N_4(x)w^h_{,x}(x_2)$$

where

$$N_1(x) = -\frac{(3x_1-x_2)x_2^2}{h^3} + \frac{6x_1x_2}{h^3}x - \frac{3(x_1+x_2)}{h^3}x^2 + \frac{2}{h^3}x^3 = \frac{(x-x_2)^2\left[h+2(x-x_1)\right]}{h^3}$$

$$N_2(x) = -\frac{x_1x_2^2}{h^2} + \frac{x_2(2x_1+x_2)}{h^2}x - \frac{x_1+2x_2}{h^2}x^2 + \frac{1}{h^2}x^3 = \frac{(x-x_1)(x-x_2)^2}{h^2}$$

$$N_3(x) = -\frac{x_1^2(x_1-3x_2)}{h^3} - \frac{6x_1x_2}{h^3}x + \frac{3(x_1+x_2)}{h^3}x^2 - \frac{2}{h^3}x^3 = \frac{(x-x_1)^2\left[h-2(x-x_2)\right]}{h^3}$$

$$N_4(x) = -\frac{x_1^2x_2}{h^2} + \frac{x_1(x_1+2x_2)}{h^2}x - \frac{2x_1+x_2}{h^2}x^2 + \frac{1}{h^2}x^3 = \frac{(x-x_1)^2(x-x_2)}{h^2}$$

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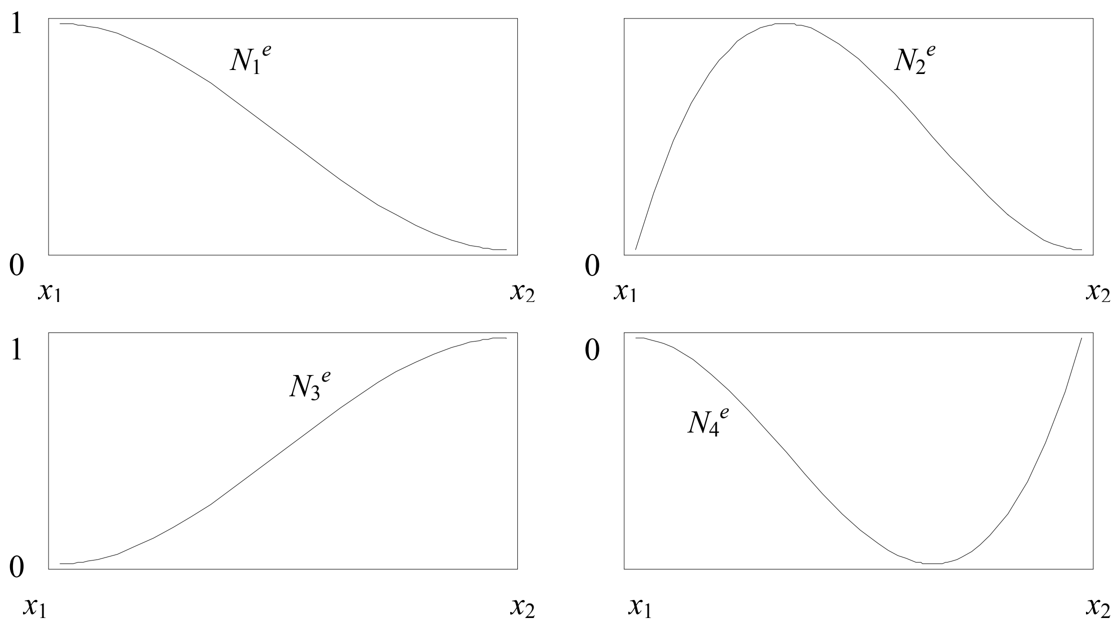
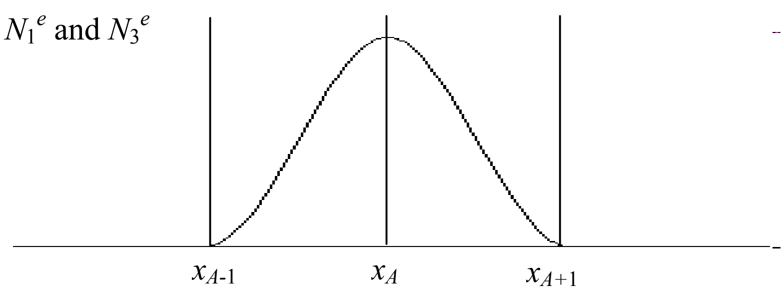
Element shape functions



Corresponding global conterparts

For N_1^e and N_3^e



For N_2^e and N_4^e