ME235A Finite Element Analysis
Fall, 2001

Solution to Problem Set # 4
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Exercise 1 (page 63)

symmetry

$$a(u,v)=\int_\Omega u_{,i}\kappa_{ij}v_{,j}d\Omega=\int_\Omega v_{,j}\kappa_{ji}u_{,i}d\Omega=a(v,u)$$

$$(u,v)=\int_\Omega uvd\Omega=\int_\Omega vud\Omega=(v,u)$$

$$(u,v)_\Gamma=\int_\mathrm{T} uvd\Gamma=\int_\Gamma vud\mathrm{T}=(v,u)_\Gamma$$

bilinearity

$$a(c_1u+c_2v,w)=\int_\Omega (c_1u+c_2v)_{,i}\kappa_{ij}w_{,j}d\Omega$$

$$=c_1\int_\Omega u_{,i}\kappa_{ij}w_{,j}d\Omega+c_2\int_\Omega v_{,i}\kappa_{ij}w_{,j}d\Omega=c_1a(u,w)+c_2a(v,w)$$

$$(c_1u+c_2v,w)=\int_\Omega (c_1u+c_2v)wd\Omega=c_1\int_\Omega uwd\Omega+c_2\int_\Omega vwd\Omega=c_1(u,w)+c_2(v,w)$$

$$(c_1u+c_2v,w)_\Gamma=\int_\Gamma (c_1u+c_2v)wd\Gamma=c_1\int_\Gamma uwd\Gamma+c_2\int_\Gamma vwd\Gamma=c_1(u,w)_\Gamma+c_2(v,w)_\Gamma$$

Exercise 2 (page 64)

for $n_{sd}=2$

$$w_{,i}\kappa_{ij}u_{,j}=w_{,1}\kappa_{11}u_{,1}+w_{,1}\kappa_{12}u_{,2}+w_{,2}\kappa_{21}u_{,1}+w_{,2}\kappa_{22}u_{,2}$$

$$=<w_{,1}\quad w_{,2}>\begin{bmatrix}\kappa_{11}&\kappa_{12}\\\kappa_{21}&\kappa_{22}\end{bmatrix}\begin{Bmatrix}u_{,1}\\u_{,2}\end{Bmatrix}=(\nabla\boldsymbol{w})^T\boldsymbol{\kappa}\nabla\boldsymbol{u}$$

for $n_{sd}=3$

$$w_{,i}\kappa_{ij}u_{,j}=w_{,1}\kappa_{11}u_{,1}+w_{,1}\kappa_{12}u_{,2}+w_{,1}\kappa_{13}u_{,3}+w_{,2}\kappa_{21}u_{,1}+w_{,2}\kappa_{22}u_{,2}+w_{,2}\kappa_{23}u_{,3}$$

$$+w_{,3}\kappa_{31}u_{,1}+w_{,3}\kappa_{32}u_{,2}+w_{,3}\kappa_{33}u_{,3}$$

$$=<w_{,1}\quad w_{,2}\quad w_{,3}>\begin{bmatrix}\kappa_{11}&\kappa_{12}&\kappa_{13}\\\kappa_{21}&\kappa_{22}&\kappa_{23}\\\kappa_{31}&\kappa_{32}&\kappa_{33}\end{bmatrix}\begin{Bmatrix}u_{,1}\\u_{,2}\\u_{,3}\end{Bmatrix}=(\nabla\boldsymbol{w})^T\boldsymbol{\kappa}\nabla\boldsymbol{u}$$

Exercise 1 (page 68)

$$-\int_\Omega w_{,i}q_id\Omega=\int_\Omega wfd\Omega+\int_{\Gamma_h}whd\Gamma$$

$$0=-\sum_{e=1}^{n_{el}}\left(\int_{\Omega^e}w_{,i}q_id\Omega+\int_{\Omega^e}wfd\Omega\right)-\int_\Gamma whd\Gamma$$

$$= \sum_{e=1}^{n_{el}} \left(\int_{\Omega^e} (wq_{i,i} - f) d\Omega + \int_{\Gamma^e} wq_i n_i d\Gamma \right) - \int_{\Gamma_h \cup \Gamma_g} whd\Gamma - \int_{\Gamma_{\text{int}}} whd\Gamma$$

$$= \sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i} - f) d\Omega + \int_{\Gamma_h \cup \Gamma_g} wq_i n_i d\Gamma + \int_{\Gamma_{\text{int}}} wq_i^+ n_i^+ d\Gamma + \int_{\Gamma_{\text{int}}} wq_i^- n_i^- d\Gamma - \int_{\Gamma_h \cup \Gamma_g} whd\Gamma - \int_{\Gamma_{iny}} whd\Gamma$$

$$= \sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i} - f) d\Omega - \int_{\Gamma_h} w(q_n + h) d\Gamma - \int_{\Gamma_{\text{int}}} w[q_n] d\Gamma$$

When w and u are globally smooth, the Euler-Lagrange equations are just

$$q_{i,i} = f \quad \text{in } \Omega$$

$$-q_n = h \quad \text{on } \Gamma_h$$