ME235A Finite Element Analysis
Fall, 2001

Solution to Problem Set #5
Nov. 19 2001

Exercise 1 (page 71)

$$q_i = -\kappa_{ij} u^h_{,j}$$

$$\boldsymbol{q}(\boldsymbol{x}) = \{q_i\} = -\mathbf{D}(\boldsymbol{x})\nabla u^h = -\mathbf{D}(\boldsymbol{x})\nabla\left(\sum_{a=1}^{n_{en}} N_a d_a^e\right)$$

$$= -\mathbf{D}(\boldsymbol{x})\sum_{a=1}^{n_{en}} \nabla N_a d_a^e = -\mathbf{D}(\boldsymbol{x})\sum_{a=1}^{n_{en}} \boldsymbol{B}_a\, d_a^e$$

Exercise 2 (page 71)

Given f: W Ø Ñ, g: G$_g$ Ø Ñ, and h: G$_h$ Ø Ñ, find u: $\overline{\Omega}$ Ø Ñ such that

$$q_{i,i} = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \Gamma_g$$

$$\lambda u - q_i n_i = h \quad \text{on } \Gamma_h$$

Weak formulation

$$0 = \int_\Omega w(q_{i,i} - f)d\Omega$$

$$= -\int_\Omega w_{,i} q_i d\Omega + \int_{\Gamma_h} w q_i n_i d\Gamma - \int_\Omega w f d\Omega$$

$$= -\int_\Omega w_{,i} q_i d\Omega + \int_{\Gamma_h} w(\lambda u - h)d\Gamma - \int_\Omega w f d\Omega$$

So, the variational equation is

$$a(w,u) + (w,\lambda u)_{\Gamma_h} = (w,f) - (w,h)_{\Gamma_h}$$

and the element stiffness matrix is given by

$$k_{ab}^e = a(N_a, N_b)_{\Omega^e} + (N_a, \lambda N_b)_{\Gamma_h^e}$$

Positive definiteness of $\boldsymbol{K}$

$$K_{PQ} = a(N_A, N_B) + (N_A, \lambda N_B)_{\Gamma_h}$$

where P = ID(A), Q = ID(B)

Let $\boldsymbol{c} = \{c_P\}$, and the associated w^h œ V^h be $w^h = \sum_{A\in\eta-\eta_g} N_A \bar{c}_A$, where $\bar{c}_A = c_P$, P = ID(A).

i. $$\boldsymbol{c}^T \boldsymbol{K}\boldsymbol{c} = \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q = \sum_{A,B\in\eta-\eta_g} \bar{c}_A \left[a(N_A, N_B) + (N_A, \lambda N_B)_{\Gamma_h}\right] \bar{c}_B$$

$$= a\left(\sum_{A\in\eta-\eta_g} N_A \bar{c}_A, \sum_{B\in\eta-\eta_g} N_B \bar{c}_B\right) + \left(\sum_{A\in\eta-\eta_g} N_A \bar{c}_A, \lambda \sum_{B\in\eta-\eta_g} N_B \bar{c}_B\right)_{\Gamma_h}$$

$$= a\left(w^h, w^h\right) + \lambda\left(w^h, w^h\right)_{\Gamma_h}$$

$$= \int_\Omega \underbrace{w^h_{,i} \kappa_{ij} w^h_{,j}}_{\geq 0} d\Omega + \lambda \int_{\Gamma_h} \underbrace{\left(w^h\right)^2}_{\geq 0} d\Gamma \geq 0$$

ii. Assume $\boldsymbol{c}^T \boldsymbol{K} \boldsymbol{c} = 0$.

$$\int_\Omega \underbrace{w^h_{,i} \kappa_{ij} w^h_{,j}}_{\geq 0} d\Omega + \lambda \int_{\Gamma_h} \underbrace{\left(w^h\right)^2}_{\geq 0} d\Gamma = 0$$

which means $w^h_{,i} \kappa_{ij} w^h_{,j} = 0$ and $w^h = 0$

Therefore $w^h = 0$, i.e., $c_p = 0$ and $\boldsymbol{c} = \boldsymbol{0}$.

Exercise 1 (page 75)

global node numbers (A)

	1	2	3	4	5	6	7	8	9	10	11	12
ID =	1	2	3	4	0	0	0	0	5	6	7	8

IEN array

element numbers (e)

local node numbers (a)	1	2	3	4	5
1	4	6	5	7	9
2	3	5	7	8	10
3	1	3	9	10	12
4	2	4	3	9	11

LM array

element numbers (e)

local node numbers (a)	1	2	3	4	5
1	4	0	0	0	5
2	3	0	0	0	6
3	1	3	5	6	8
4	2	4	3	5	7

Exercise 8 (page 107)

i. $$-\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w f d\Omega + \int_{\Gamma_h} w h d\Gamma + \int_{\Gamma_g} w \bar{h} d\Gamma$$

(the bar over $\bar{h}$ is to distinguish between the known (prescribed) flux on Γ_h and the unknown flux on Γ_g.)

Integrating by parts,

$$\int_\Omega w q_{i,i}\,d\Omega - \int_{\Gamma_h} w q_i n_i\,d\Gamma - \int_{\Gamma_g} w q_i n_i\,d\Gamma = \int_\Omega w f\,d\Omega + \int_{\Gamma_h} w h\,d\Gamma + \int_{\Gamma_g} w \bar{h}\,d\Gamma$$

$$0 = \int_\Omega w\left(q_{i,i} - f\right)d\Omega - \int_{\Gamma_h} w\left(q_i n_i + h\right)d\Gamma - \int_{\Gamma_g} w\left(q_i n_i + \bar{h}\right)d\Gamma$$

Let $\alpha = q_{i,i} - f$, $\beta = q_i n_i + h$, $\gamma = q_i n_i + \bar{h}$.

Following the text, choose $w = \alpha\varphi$, where $\varphi = 0$ on Γ, $\varphi > 0$ in Ω and φ is smooth ($\varphi \in \mathcal{V}$).

Then, $0 = \int_\Omega \underbrace{\varphi}_{>0}\underbrace{\alpha^2}_{\geq 0} d\Omega + 0$ so that $\alpha = 0$ in Ω.

Now, choose $w = \beta\phi$, where $\phi = 0$ on Γ_g, $\phi > 0$ on Γ_h, and ϕ is smooth ($\phi \in \mathcal{V}$).

Then, $0 = \int_{\Gamma_h} \underbrace{\phi}_{>0}\underbrace{\beta^2}_{\geq 0} d\Gamma + 0$ so that $\beta = 0$ on Γ_h.

Now, choose $w = \gamma\psi$, where $\psi = 0$ on Γ_h, $\psi > 0$ on Γ_g, and ψ is smooth ($\psi \in \mathcal{V}$).

Then, $0 = \int_{\Gamma_g} \underbrace{\psi}_{>0}\underbrace{\gamma^2}_{\geq 0} d\Gamma$ so that $\gamma = 0$ on Γ_g.

The Euler-Lagrange equations are

$$q_{i,i} = f \text{ in } \Omega$$

$$-q_i n_i = h \text{ on } \Gamma_h$$

$$-q_i n_i = \bar{h} \text{ on } \Gamma_g$$

ii. The variational equation is

$$a(w,u) = (w,f) + (w,h)_{\Gamma_h} + (w,\bar{h})_{\Gamma_g}$$

The Galerkin formulation

$$u^h = v^h + g^h$$

$$a\left(w^h, v^h\right) - \left(w^h, \bar{h}^h\right)_{\Gamma_g} = \left(w^h, f\right) + \left(w^h, h\right)_{\Gamma_h} - a\left(w^h, g^h\right)$$

Let $w^h = \sum_{A\in\eta} N_A c_A$, $v^h = \sum_{A\in\eta-\eta_g} N_A d_A$, $h^h = \sum_{A\in\eta_g} N_A \bar{h}_A$, and $g^h = \sum_{A\in\eta_g} N_A g_A$

(note that w^h is over *all* the nodes)

Then, for $\forall A \in \eta$

$$\sum_{B\in\eta-\eta_g} a\left(N_A, N_B\right)d_B - \sum_{B\in\eta_g}\left(N_A, N_B\right)_{\Gamma_g}\bar{h}_B = \left(N_A, f\right) + \left(N_A, h\right)_{\Gamma_h} - \sum_{B\in\eta_g} a\left(N_A, N_B\right)g_B$$

The matrix formulation is

$$\boldsymbol{Kd} - \overline{\boldsymbol{K}}\,\boldsymbol{h} = \boldsymbol{F}$$

where

$\boldsymbol{K} = \lfloor K_{PQ} \rfloor$,	$K_{PQ} = a\left(N_A, N_B\right)$	($n_{np} \times n_{eq}$)
$\overline{\boldsymbol{K}} = \lfloor \overline{K}_{PQ} \rfloor$,	$\overline{K}_{PQ} = \left(N_A, N_B\right)_{\Gamma_g}$	($n_{np} \times (n_{np} - n_{eq})$)
$\boldsymbol{d} = \{d_Q\}$		($n_{eq} \times 1$)
$\boldsymbol{h} = \{\bar{h}_Q\}$		($(n_{np} - n_{eq}) \times 1$)

$$\boldsymbol{F} = \{F_P\}, \quad F_P = (N_A, f) + (N_A, h)_{\Gamma_h} - \sum_{B\in\eta_g} a(N_A, N_B) g_B \qquad (n_{np} \text{ µ } 1)$$

$P = \mathrm{ID}(A)$, $Q = \mathrm{ID}(B)$

Or, we can write (check to see that the matrix dimensions allow this, letting us solve for both $\boldsymbol{d}$ and $\boldsymbol{h}$)

$$\underbrace{\begin{bmatrix}\boldsymbol{K} & \overline{\boldsymbol{K}}\end{bmatrix}}_{n_{np}\times n_{np}} \underbrace{\begin{Bmatrix}\boldsymbol{d}\\ \boldsymbol{h}\end{Bmatrix}}_{n_{np}\times 1} = \underbrace{\boldsymbol{F}}_{n_{np}\times 1}$$

Exercise 1 (page 81)

symmetry

$$a(\boldsymbol{w},\boldsymbol{u}) = \int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)} d\Omega = \int_\Omega u_{(k,l)} c_{klij} w_{(i,j)} d\Omega = a(\boldsymbol{u},\boldsymbol{w})$$

$$(\boldsymbol{w},\boldsymbol{f}) = \int_\Omega w_i f_i d\Omega = \int_\Omega f_i w_i d\Omega = (\boldsymbol{f},\boldsymbol{w})$$

$$(\boldsymbol{w},\boldsymbol{h})_{\Gamma_h} = \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i d\Gamma \right) = \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} h_i w_i d\Gamma \right) = (\boldsymbol{h},\boldsymbol{w})_{\Gamma_h}$$

bilinearity

$$a(c_1\,\boldsymbol{u} + c_2\,\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (c_1\,\boldsymbol{u} + c_2\,\boldsymbol{v})_{(i,j)} c_{ijkl} w_{(k,l)} d\Omega$$

$$(c_1\,\boldsymbol{u} + c_2\,\boldsymbol{v})_{(i,j)} = \frac{1}{2}\left((c_1 u_i + c_2 v_i)_{,j} + (c_1 u_j + c_2 v_j)_{,i} \right) = c_1 u_{(i,j)} + c_2 v_{(i,j)}$$

$$\begin{aligned} a(c_1\,\boldsymbol{u} + c_2\,\boldsymbol{v},\boldsymbol{w}) &= \int_\Omega \left(c_1 u_{(i,j)} + c_2 v_{(i,j)}\right) c_{ijkl} w_{(k,l)} d\Omega \\ &= c_1 \int_\Omega u_{(i,j)} c_{ijkl} w_{(i,j)} d\Omega + c_2 \int_\Omega v_{(i,j)} c_{ijkl} w_{(i,j)} d\Omega \\ &= c_1(\boldsymbol{u},\boldsymbol{w}) + c_2(\boldsymbol{v},\boldsymbol{w}) \end{aligned}$$

$$(c_1\,\boldsymbol{u} + c_2\,\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (c_1 u + c_2 v)_i w_i d\Omega = c_1 \int_\Omega u_i w_i d\Omega + c_2 \int_\Omega v_i w_i d\Omega = c_1(\boldsymbol{u},\boldsymbol{w}) + c_2(\boldsymbol{v},\boldsymbol{w})$$

$$\begin{aligned} (c_1\,\boldsymbol{u} + c_2\,\boldsymbol{v},\boldsymbol{w})_{\Gamma_h} &= \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} (c_1 u_i + c_2 v_i) w_i d\Gamma \right) = c_1 \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} u_i w_i d\Gamma \right) + c_2 \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} v_i w_i d\Gamma \right) \\ &= c_1(\boldsymbol{u},\boldsymbol{w})_{\Gamma_h} + c_2(\boldsymbol{v},\boldsymbol{w})_{\Gamma_h} \end{aligned}$$

Exercise 2 (page 82)

$$\begin{aligned} w_{(i,j)} c_{ijkl} u_{(k,l)} = {} & w_{(1,1)} c_{1111} u_{(1,1)} + w_{(1,1)} c_{1122} u_{(2,2)} + w_{(1,1)} c_{1112} u_{(1,2)} + w_{(1,1)} c_{1121} u_{(2,1)} \\ & + w_{(2,2)} c_{2211} u_{(1,1)} + w_{(2,2)} c_{1122} u_{(2,2)} + w_{(2,2)} c_{2212} u_{(1,2)} + w_{(2,2)} c_{2221} u_{(2,1)} \\ & + w_{(1,2)} c_{1211} u_{(1,1)} + w_{(1,2)} c_{1122} u_{(2,2)} + w_{(1,2)} c_{1212} u_{(1,2)} + w_{(1,2)} c_{1221} u_{(2,1)} \\ & + w_{(2,1)} c_{2111} u_{(1,1)} + w_{(2,1)} c_{2122} u_{(2,2)} + w_{(2,1)} c_{2112} u_{(1,2)} + w_{(2,1)} c_{2121} u_{(2,1)} \end{aligned}$$

$$\begin{aligned}&= w_{(1,1)}\left(c_{1111}u_{(1,1)} + c_{1122}u_{(2,2)} + 2c_{1112}u_{(1,2)}\right)\\ &+ w_{(2,2)}\left(c_{2211}u_{1,1} + c_{2222}u_{(2,2)} + 2c_{2212}u_{(1,2)}\right)\\ &+ 2w_{(1,2)}\left(c_{1211}u_{(1,1)} + c_{1222}u_{(2,2)} + 2c_{1212}u_{(1,2)}\right)\end{aligned}$$

Since

$$2w_{(1,2)} = w_{1,2} + w_{2,1}\,,\ 2u_{(1,2)} = u_{1,2} + u_{2,1}$$

and $c_{1211} = c_{1112}$, $c_{2211} = c_{1122}$, $c_{1222} = c_{1212}$,

$$w_{(i,j)}c_{ijkl}u_{(k,l)} = \left\{w_{1,1} \quad w_{2,2} \quad w_{1,2} + w_{2,1}\right\}\begin{bmatrix} c_{1111} & c_{1122} & c_{1112}\\ c_{1122} & c_{2222} & c_{2212}\\ c_{1112} & c_{2212} & c_{1212}\end{bmatrix}\begin{Bmatrix} u_{1,1}\\ u_{2,2}\\ u_{1,2} + u_{2,1}\end{Bmatrix}$$

$$= \boldsymbol{\varepsilon}(\boldsymbol{w})^T\ \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u})$$

Exercise 3 (page 82)

$$\boldsymbol{\varepsilon}(\boldsymbol{w})^T\ \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \begin{Bmatrix} w_{1,1}\\ w_{2,2}\\ w_{3,3}\\ w_{2,3}+w_{3,2}\\ w_{1,3}+w_{3,1}\\ w_{1,2}+w_{2,1}\end{Bmatrix}^T \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16}\\ \bullet & D_{22} & D_{23} & D_{24} & D_{25} & D_{26}\\ \bullet & \bullet & D_{33} & D_{34} & D_{35} & D_{36}\\ \bullet & \bullet & \bullet & D_{44} & D_{45} & D_{46}\\ \bullet & \bullet & \bullet & \bullet & D_{55} & D_{56}\\ \bullet & \bullet & \bullet & \bullet & \bullet & D_{66}\end{bmatrix}\begin{Bmatrix} u_{1,1}\\ u_{2,2}\\ u_{3,3}\\ u_{2,3}+u_{3,2}\\ u_{1,3}+u_{3,1}\\ u_{1,2}+u_{2,1}\end{Bmatrix}$$

(The dots indicate symmetric elements)

$D_{IJ} = c_{ijkl}$

I / J	i / k	j / l
1	1	1
2	2	2
3	3	3
4	2	3
4	3	2
5	1	3
5	3	1
6	1	2
6	2	1

Exercise 4 (page 82)

$$
\begin{aligned}
\sigma_{ij} &= c_{ijkl} u_{(k,l)} = \frac{1}{2} c_{ijkl} \left(u_{k,l} + u_{l,k} \right) \\
&= c_{ij11} u_{1,1} + c_{ij22} u_{2,2} + c_{ij12} \left(u_{1,2} + u_{2,1} \right)
\end{aligned}
$$

$$
\boldsymbol{D\varepsilon}(\boldsymbol{u}) = \begin{bmatrix} c_{1111} & c_{1122} & c_{1112} \\ c_{1122} & c_{2222} & c_{2212} \\ c_{1112} & c_{2212} & c_{1212} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix}
$$

$$
= \begin{Bmatrix} c_{1111} u_{1,1} + c_{1122} u_{2,2} + c_{1112} \left(u_{1,2} + u_{2,1} \right) \\ c_{2211} u_{1,1} + c_{2222} u_{2,2} + c_{2212} \left(u_{1,2} + u_{2,1} \right) \\ c_{1211} u_{1,1} + c_{1222} u_{2,2} + c_{1212} \left(u_{1,2} + u_{2,1} \right) \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \boldsymbol{\sigma}
$$

And similarly for $n_{sd} = 3$.

Exercise 5 (page 83)

$$
c_{ijkl} = \mu(\boldsymbol{x}) \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + \lambda(\boldsymbol{x}) \delta_{ij} \delta_{kl}
$$

for $n_{sd} = 2$

$D_{11} = c_{1111} = 2m + l$
$D_{12} = c_{1122} = l$
$D_{13} = c_{1112} = 0$
$D_{22} = c_{2222} = 2m + l$
$D_{23} = c_{2212} = 0$
$D_{33} = c_{1212} = m$

So that

$$
\boldsymbol{D} = \begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \bullet & 2\mu + \lambda & 0 \\ \bullet & \bullet & \mu \end{bmatrix}
$$

for $n_{sd} = 3$

$D_{11} = c_{1111} = 2m + l$
$D_{12} = c_{1122} = l$
$D_{13} = c_{1133} = l$
$D_{22} = c_{2222} = 2m + l$
$D_{23} = c_{2233} = l$
$D_{33} = c_{3333} = 2m + l$
$D_{44} = c_{2323} = m$
$D_{55} = c_{1313} = m$
$D_{66} = c_{1212} = m$
$D_{14} = D_{15} = D_{16} = D_{24} = D_{25} = D_{26} = D_{34} = D_{35} = D_{36} = D_{45} = D_{46} = D_{56} = 0$

So that,

$$\boldsymbol{D} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \bullet & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \bullet & \bullet & 2\mu+\lambda & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \mu & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \mu & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \mu \end{bmatrix}$$

Exercise 1 (page 87)

$$K_{PQ} = a\left(N_A\,\boldsymbol{e}_i\,, N_B\,\boldsymbol{e}_j\right) = \int_\Omega \boldsymbol{\varepsilon}\left(N_A\,\boldsymbol{e}_i\right)^T\,\boldsymbol{D}\boldsymbol{\varepsilon}\left(N_B\,\boldsymbol{e}_j\right)d\Omega$$

for $n_{sd} = 2$,

$$\boldsymbol{\varepsilon}\left(N_A\,\boldsymbol{e}_i\right) = \begin{Bmatrix} N_{A,1}\delta_{i1} \\ N_{A,2}\delta_{i2} \\ N_{A,2}\delta_{i1} + N_{A,1}\delta_{i2} \end{Bmatrix} = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \begin{Bmatrix} \delta_{i1} \\ \delta_{i2} \end{Bmatrix}$$

since, $\boldsymbol{e}_i = \begin{Bmatrix} \delta_{i1} \\ \delta_{i2} \end{Bmatrix}$

$$\boldsymbol{\varepsilon}\left(N_A\,\boldsymbol{e}_i\right) = \boldsymbol{B}_A\,\boldsymbol{e}_i$$

where

$$B = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix}$$

for $n_{sd} = 3$,

$$\boldsymbol{\varepsilon}\left(N_A\,\boldsymbol{e}_i\right) = \begin{Bmatrix} N_{A,1}\delta_{i1} \\ N_{A,2}\delta_{i2} \\ N_{A,3}\delta_{i3} \\ N_{A,3}\delta_{i2} + N_{A,2}\delta_{i3} \\ N_{A,3}\delta_{i1} + N_{A,1}\delta_{i3} \\ N_{A,2}\delta_{i1} + N_{A,1}\delta_{i2} \end{Bmatrix} = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix} \begin{Bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{Bmatrix} = \boldsymbol{B}_A\,\boldsymbol{e}_i$$

where

$$\boldsymbol{B}_A = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix}$$

Exercise 1 (page 91)

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{u}^h)$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^h) = \begin{Bmatrix} u_{1,1}^h \\ u_{2,2}^h \\ u_{1,2}^h + u_{2,1}^h \end{Bmatrix} = \begin{Bmatrix} \sum_{a=1}^{n_{en}} N_{a,1}(\boldsymbol{x}) u_1^h(\boldsymbol{x}_a^e) \\ \sum_{a=1}^{n_{en}} N_{a,2}(\boldsymbol{x}) u_2^h(\boldsymbol{x}_a^e) \\ \sum_{a=1}^{n_{en}} N_{a,2}(\boldsymbol{x}) u_1^h(\boldsymbol{x}_a^e) + \sum_{a=1}^{n_{en}} N_{a,1}(\boldsymbol{x}) u_2^h(\boldsymbol{x}_a^e) \end{Bmatrix}$$

$$= \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(\boldsymbol{x}) \begin{Bmatrix} d_{1a}^e \\ d_{2a}^e \end{Bmatrix} = \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(\boldsymbol{x}) \boldsymbol{d}_a^e = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{d}^e$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{u}^h) = \boldsymbol{D}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{d}^e = \boldsymbol{D}(\boldsymbol{x}) \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(\boldsymbol{x}) \boldsymbol{d}_a^e$$

Exercise 1 (page98)

ID array

Global node numbers (A) across columns; Golbal degrees of freedom (i) down rows:

	1	2	3	4	5	6	7	8
1	1	3	0	5	7	9	0	12
2	2	4	0	6	8	10	11	13

IEN array

Element numbers (e) across columns; Local node numbers (a) down rows:

	1	2	3	4
1	1	1	4	8
2	2	3	3	6
3	8	4	5	5
4	7	2	6	7

LM array

Local equation numbers (p) \ Element number (e)	1	2	3	4
1	1	1	5	12
2	2	2	6	13
3	3	0	0	9
4	4	0	0	10
5	12	5	7	7
6	13	6	8	8
7	0	3	9	0
8	11	4	10	11