ME235A Finite Element Analysis
Fall, 2001

Solution to Problem Set #6
Dec. 12 2001

Exercise 6 (page 84)

The weak form of the problem is

$$\int_{\Omega} w_{(i,j)}\sigma_{ij}\, d\Omega = \int_{\Omega} w_i f_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma$$

Separating into element domains and integrating by parts,

$$\sum_{e=1}^{n_{el}} \int_{\Omega} w_{i,j}\sigma_{ij}\, d\Omega = \sum_{e=1}^{n_{el}} \int_{\Omega} w_i f_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma$$

$$\sum_{e=1}^{n_{el}} \left(-\int_{\Omega^e} w_i \sigma_{ij,j}\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}^e} w_i \sigma_{ij} n_j\, d\Gamma \right) = \sum_{e=1}^{n_{el}} \int_{\Omega} w_i f_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma$$

Since

$$\sum_{e=1}^{n_{el}} \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}^e} w_i \sigma_{ij} n_j\, d\Gamma = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i \sigma_{in}\, d\Gamma + \int_{\Gamma_{\boldsymbol{int}}} w_i [\sigma_{in}]\, d\Gamma$$

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w_i (\sigma_{ij,j} + f_i)\, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i (\sigma_{in} - h_i)\, d\Gamma - \int_{\Gamma_{\boldsymbol{int}}} w_i [\sigma_{in}]\, d\Gamma$$

For globally smooth w_i and u_i, the Euler-Lagrange equations are

$$\sigma_{ij} + f_i = 0 \quad \text{on } \Omega$$

$$\sigma_{in} = h_i \quad \text{on } \Gamma_{h_i}$$

Exercise 1 (page 114)

From

$$\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}, \qquad \begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix}$$

Solving for the α's and β's,

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} x_1^e + x_2^e + x_3^e + x_4^e \\ -x_1^e + x_2^e + x_3^e - x_4^e \\ -x_1^e - x_2^e + x_3^e + x_4^e \\ x_1^e - x_2^e + x_3^e - x_4^e \end{Bmatrix}, \qquad \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} y_1^e + y_2^e + y_3^e + y_4^e \\ -y_1^e + y_2^e + y_3^e - y_4^e \\ -y_1^e - y_2^e + y_3^e + y_4^e \\ y_1^e - y_2^e + y_3^e - y_4^e \end{Bmatrix}$$

Substituting the α's and β's and rearranging, we have

$$x(\xi,\eta)=\frac{1}{4}\left(x_1^e+x_2^e+x_3^e+x_4^e\right)+\frac{1}{4}\xi\left(-x_1^e+x_2^e+x_3^e-x_4^e\right)+\frac{1}{4}\eta\left(-x_1^e-x_2^e+x_3^e+x_4^e\right)$$
$$+\frac{1}{4}\xi\eta\left(x_1^e-x_2^e+x_3^e-x_4^e\right)$$
$$=\frac{1}{4}(1-\xi)(1-\eta)x_1^e+\frac{1}{4}(1+\xi)(1-\eta)x_2^e+\frac{1}{4}(1+\xi)(1+\eta)x_3^e+\frac{1}{4}(1-\xi)(1+\eta)x_4^e$$
$$y(\xi,\eta)=\frac{1}{4}\left(y_1^e+y_2^e+y_3^e+y_4^e\right)+\frac{1}{4}\xi\left(-y_1^e+y_2^e+y_3^e-y_4^e\right)+\frac{1}{4}\eta\left(-y_1^e-y_2^e+y_3^e+y_4^e\right)$$
$$+\frac{1}{4}\xi\eta\left(y_1^e-y_2^e+y_3^e-y_4^e\right)$$
$$=\frac{1}{4}(1-\xi)(1-\eta)y_1^e+\frac{1}{4}(1+\xi)(1-\eta)y_2^e+\frac{1}{4}(1+\xi)(1+\eta)y_3^e+\frac{1}{4}(1-\xi)(1+\eta)y_4^e$$

Comparing with

$$x(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)x_a^e$$
$$y(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)y_a^e$$

We can see that

$$N_a(\xi)=\frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$$

Exercise 1 (page 123)

$$\boldsymbol{x}=\sum_{a=1}^{3}N_a{}'\,\boldsymbol{x}_a^e$$

where

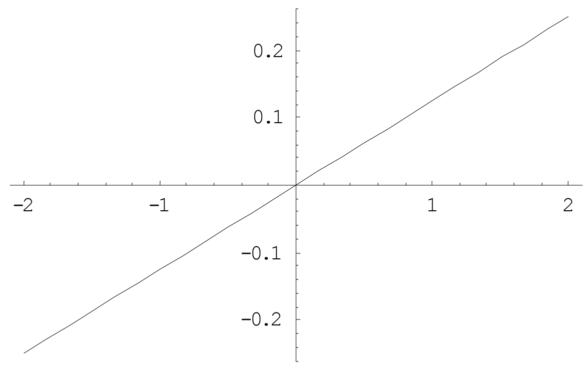
$$N_1{}'=\frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2{}'=\frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3{}'=\frac{1}{2}(1+\eta)$$

So, we have

$$x_{,\xi}=\frac{1}{4}(1-\eta)\left(-x_1^e+x_2^e\right)$$
$$x_{,\eta}=-\frac{1}{4}(1-\xi)x_1^e-\frac{1}{4}(1+\xi)x_2^e+\frac{1}{2}x_3^e$$
$$y_{,\xi}=\frac{1}{4}(1-\eta)\left(-y_1^e+y_2^e\right)$$
$$y_{,\eta}=-\frac{1}{4}(1-\xi)y_1^e-\frac{1}{4}(1+\xi)y_2^e+\frac{1}{2}y_3^e$$

Taking $y_1^e = 0$, $x_2^e = 0$, $y_2^e = 1$, $x_3^e = 0$, $y_3^e = 0$ and $\xi = \eta = 0$,

$$j = \det\begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix} = \det\begin{bmatrix} -\frac{1}{4}x_1^e & -\frac{1}{4}x_1^e \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = \frac{1}{8}x_1^e$$



Notice that when $x_1^e < 0$, the nodes are no longer in counterclockwise order, and the negative Jacobian reflects that fact.

Exercise 1 (page 128)

$$l_a^{n_{en}-1}(\xi) = \frac{\prod_{\substack{b=1 \\ b \neq a}}^{n_{en}} (\xi - \xi_b)}{\prod_{\substack{b=1 \\ b \neq a}}^{n_{en}} (\xi_a - \xi_b)}$$

when $n_{en} = 4$

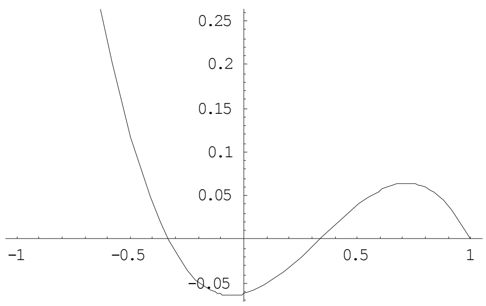
$$N_1 = l_1^3 = \frac{(\xi + 1/3)(\xi - 1/3)(\xi - 1)}{(-1 + 1/3)(-1 - 1/3)(-1 - 1)} = -\frac{9}{16}(\xi + 1/3)(\xi - 1/3)(\xi - 1)$$

$$N_2 = l_2^3 = \frac{(\xi + 1)(\xi - 1/3)(\xi - 1)}{(-1/3 + 1)(-1/3 - 1/3)(-1/3 - 1)} = \frac{27}{16}(\xi + 1)(\xi - 1/3)(\xi - 1)$$

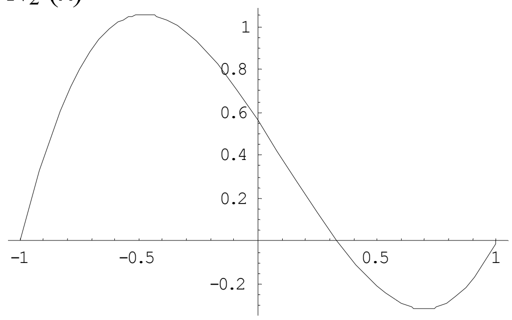
$$N_3 = l_3^3 = \frac{(\xi + 1)(\xi + 1/3)(\xi - 1)}{(1/3 + 1)(1/3 + 1/3)(1/3 - 1)} = -\frac{27}{16}(\xi + 1)(\xi + 1/3)(\xi - 1)$$

$$N_4 = l_4^3 = \frac{(\xi + 1)(\xi + 1/3)(\xi - 1/3)}{(1 + 1)(1 + 1/3)(1 - 1/3)} = \frac{9}{16}(\xi + 1)(\xi + 1/3)(\xi - 1/3)$$

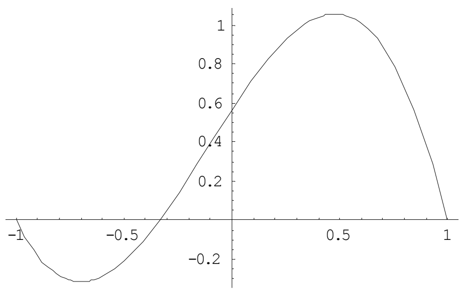
N_1 (x)



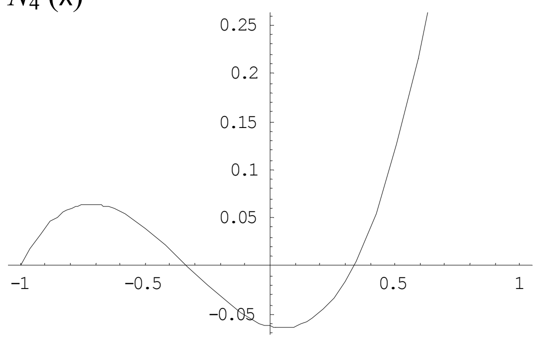
N_2 (x)



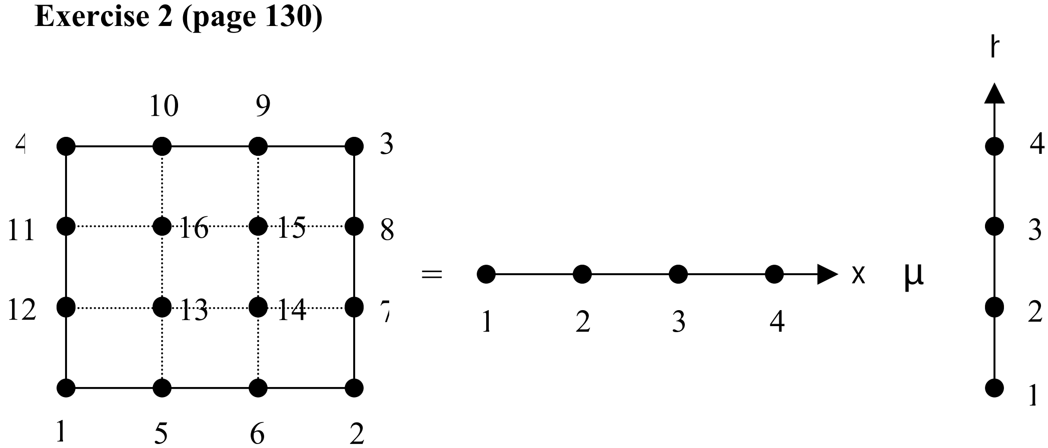
N_3 (x)



N_4 (x)



Exercise 2 (page 130)



$$N_1 = l_1^3(\xi) l_1^3(\eta) = \frac{81}{256}\left(\xi + \tfrac{1}{3}\right)\left(\xi - \tfrac{1}{3}\right)(\xi - 1)\left(\eta + \tfrac{1}{3}\right)\left(\eta - \tfrac{1}{3}\right)(\eta - 1)$$

$$N_2 = l_4^3(\xi) l_1^3(\eta) = -\frac{81}{256}(\xi + 1)\left(\xi + \tfrac{1}{3}\right)\left(\xi - \tfrac{1}{3}\right)\left(\eta + \tfrac{1}{3}\right)\left(\eta - \tfrac{1}{3}\right)(\eta - 1)$$

$$N_3 = l_4^3(\xi) l_4^3(\eta) = \frac{81}{256}(\xi + 1)\left(\xi + \tfrac{1}{3}\right)\left(\xi - \tfrac{1}{3}\right)(\eta + 1)\left(\eta + \tfrac{1}{3}\right)\left(\eta - \tfrac{1}{3}\right)$$

$$N_4 = l_1^3(\xi) l_4^3(\eta) = -\frac{81}{256}\left(\xi + \tfrac{1}{3}\right)\left(\xi - \tfrac{1}{3}\right)(\xi - 1)(\eta + 1)\left(\eta + \tfrac{1}{3}\right)\left(\eta - \tfrac{1}{3}\right)$$

$$N_5 = l_2^3(\xi) l_1^3(\eta) = -\frac{243}{256}(\xi+1)(\xi-1/3)(\xi-1)(\eta+1/3)(\eta-1/3)(\eta-1)$$

$$N_6 = l_3^3(\xi) l_1^3(\eta) = \frac{243}{256}(\xi+1)(\xi+1/3)(\xi-1)(\eta+1/3)(\eta-1/3)(\eta-1)$$

$$N_7 = l_4^3(\xi) l_2^3(\eta) = \frac{243}{256}(\xi+1)(\xi+1/3)(\xi-1/3)(\eta+1)(\eta-1/3)(\eta-1)$$

$$N_8 = l_4^3(\xi) l_3^3(\eta) = -\frac{243}{256}(\xi+1)(\xi+1/3)(\xi-1/3)(\eta+1)(\eta+1/3)(\eta-1)$$

$$N_9 = l_3^3(\xi) l_4^3(\eta) = -\frac{243}{256}(\xi+1)(\xi+1/3)(\xi-1)(\eta+1)(\eta+1/3)(\eta-1/3)$$

$$N_{10} = l_2^3(\xi) l_4^3(\eta) = \frac{243}{256}(\xi+1)(\xi-1/3)(\xi-1)(\eta+1)(\eta+1/3)(\eta-1/3)$$

$$N_{11} = l_1^3(\xi) l_3^3(\eta) = \frac{243}{256}(\xi+1/3)(\xi-1/3)(\xi-1)(\eta+1)(\eta+1/3)(\eta-1)$$

$$N_{12} = l_1^3(\xi) l_2^3(\eta) = -\frac{243}{256}(\xi+1/3)(\xi-1/3)(\xi-1)(\eta+1/3)(\eta-1/3)(\eta-1)$$

$$N_{13} = l_2^3(\xi) l_2^3(\eta) = \frac{729}{256}(\xi+1/3)(\xi-1/3)(\xi-1)(\eta+1/3)(\eta-1/3)(\eta-1)$$

$$N_{14} = l_3^3(\xi) l_2^3(\eta) = -\frac{729}{256}(\xi+1)(\xi+1/3)(\xi-1)(\eta+1/3)(\eta-1/3)(\eta-1)$$

$$N_{15} = l_3^3(\xi) l_3^3(\eta) = \frac{729}{256}(\xi+1)(\xi+1/3)(\xi-1)(\eta+1)(\eta+1/3)(\eta-1)$$

$$N_{16} = l_2^3(\xi) l_3^3(\eta) = -\frac{729}{256}(\xi+1/3)(\xi-1/3)(\xi-1)(\eta+1)(\eta+1/3)(\eta-1)$$

Exercise 4 (page 130)

For the x coordinate,

$$l_1^2(\xi) = \frac{1}{2}\xi(\xi-1)$$

$$l_2^2(\xi) = 1-\xi^2$$

$$l_3^2(\xi) = \frac{1}{2}\xi(\xi+1)$$

For the h coordinate,

$$l_1^1(\eta) = \frac{1}{2}(1-\eta)$$

$$l_2^1(\eta) = \frac{1}{2}(1+\eta)$$

Using the above, the shape functions for the 6-node element are

$$N_1(\xi,\eta) = l_1^3(\xi) l_1^1(\eta) = -\frac{1}{4}\xi(\xi-1)(\eta-1)$$

$$N_2(\xi,\eta) = l_3^3(\xi) l_1^1(\eta) = -\frac{1}{4}\xi(\xi+1)(\eta-1)$$

$$N_3(\xi,\eta) = l_3^3(\xi) l_2^1(\eta) = \frac{1}{4}\xi(\xi+1)(\eta+1)$$

$$N_4(\xi,\eta) = l_1^3(\xi) l_2^1(\eta) = \frac{1}{4}\xi(\xi-1)(\eta+1)$$

$$N_5(\xi,\eta) = l_2^3(\xi) l_1^1(\eta) = -\frac{1}{2}(1-\xi^2)(\eta-1)$$

$$N_6(\xi,\eta) = l_2^3(\xi) l_2^1(\eta) = \frac{1}{2}(1-\xi^2)(\eta+1)$$

Exercise 3 (page 130)

Look at Figure 3.7.6 for the 27-node 3-dimensional element.

$$l_1^2(\xi) = \frac{1}{2}\xi(\xi-1) \qquad l_1^2(\eta) = \frac{1}{2}\eta(\eta-1) \qquad l_1^2(\zeta) = \frac{1}{2}\zeta(\zeta-1)$$

$$l_2^2(\xi) = 1-\xi^2 \qquad l_2^2(\eta) = 1-\eta^2 \qquad l_2^2(\zeta) = 1-\zeta^2$$

$$l_3^2(\xi) = \frac{1}{2}\xi(\xi+1) \qquad l_3^2(\eta) = \frac{1}{2}\eta(\eta+1) \qquad l_3^2(\zeta) = \frac{1}{2}\zeta(\zeta+1)$$

So, the shape functions are

$$N_1 = l_1^2(\xi) l_1^2(\eta) l_1^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi-1)(\eta-1)(\zeta-1)$$

$$N_2 = l_3^2(\xi) l_1^2(\eta) l_1^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi+1)(\eta-1)(\zeta-1)$$

$$N_3 = l_3^2(\xi) l_3^2(\eta) l_1^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi+1)(\eta+1)(\zeta-1)$$

$$N_4 = l_1^2(\xi) l_3^2(\eta) l_1^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi-1)(\eta+1)(\zeta-1)$$

$$N_5 = l_1^2(\xi) l_1^2(\eta) l_3^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi-1)(\eta-1)(\zeta+1)$$

$$N_6 = l_3^2(\xi) l_1^2(\eta) l_3^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi+1)(\eta-1)(\zeta+1)$$

$$N_7 = l_3^2(\xi) l_3^2(\eta) l_3^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi+1)(\eta+1)(\zeta+1)$$

$$N_8 = l_1^2(\xi) l_3^2(\eta) l_3^2(\zeta) = \frac{1}{8}\xi\eta\zeta(\xi-1)(\eta+1)(\zeta+1)$$

$$N_9 = l_2^2(\xi) l_1^2(\eta) l_1^2(\zeta) = \frac{1}{4}\eta\zeta(1-\xi^2)(\eta-1)(\zeta-1)$$

$$N_{10} = l_3^2(\xi) l_2^2(\eta) l_1^2(\zeta) = \frac{1}{4}\xi\zeta(\xi+1)(1-\eta^2)(\zeta-1)$$

$$N_{11} = l_2^2(\xi) l_3^2(\eta) l_1^2(\zeta) = \frac{1}{4}\eta\zeta(1-\xi^2)(\eta+1)(\zeta-1)$$

$$N_{12} = l_1^2(\xi)l_2^2(\eta)l_1^2(\zeta) = \frac{1}{4}\xi\zeta(\xi-1)(1-\eta^2)(\zeta-1)$$
$$N_{13} = l_2^2(\xi)l_1^2(\eta)l_3^2(\zeta) = \frac{1}{4}\eta\zeta(1-\xi^2)(\eta-1)(\zeta+1)$$
$$N_{14} = l_3^2(\xi)l_2^2(\eta)l_3^2(\zeta) = \frac{1}{4}\xi\zeta(\xi+1)(1-\eta^2)(\zeta+1)$$
$$N_{15} = l_2^2(\xi)l_3^2(\eta)l_3^2(\zeta) = \frac{1}{4}\eta\zeta(1-\xi^2)(\eta+1)(\zeta+1)$$
$$N_{16} = l_1^2(\xi)l_2^2(\eta)l_3^2(\zeta) = \frac{1}{4}\xi\zeta(\xi-1)(1-\eta^2)(\zeta+1)$$
$$N_{17} = l_1^2(\xi)l_1^2(\eta)l_2^2(\zeta) = \frac{1}{4}\xi\eta(\xi-1)(\eta-1)(1-\zeta^2)$$
$$N_{18} = l_3^2(\xi)l_1^2(\eta)l_2^2(\zeta) = \frac{1}{4}\xi\eta(\xi+1)(\eta-1)(1-\zeta^2)$$
$$N_{19} = l_3^2(\xi)l_3^2(\eta)l_2^2(\zeta) = \frac{1}{4}\xi\eta(\xi+1)(\eta+1)(1-\zeta^2)$$
$$N_{20} = l_1^2(\xi)l_3^2(\eta)l_2^2(\zeta) = \frac{1}{8}\xi\eta(\xi-1)(\eta+1)(1-\zeta^2)$$
$$N_{21} = l_2^2(\xi)l_2^2(\eta)l_1^2(\zeta) = \frac{1}{2}\zeta(1-\xi^2)(1-\eta^2)(\zeta-1)$$
$$N_{22} = l_2^2(\xi)l_2^2(\eta)l_3^2(\zeta) = \frac{1}{2}\zeta(1-\xi^2)(1-\eta^2)(\zeta+1)$$
$$N_{23} = l_2^2(\xi)l_1^2(\eta)l_2^2(\zeta) = \frac{1}{2}\eta(1-\xi^2)(\eta-1)(1-\zeta^2)$$
$$N_{24} = l_2^2(\xi)l_3^2(\eta)l_2^2(\zeta) = \frac{1}{2}\eta(1-\xi^2)(\eta+1)(1-\zeta^2)$$
$$N_{25} = l_1^2(\xi)l_2^2(\eta)l_2^2(\zeta) = \frac{1}{2}\xi(\xi-1)(1-\eta^2)(1-\zeta^2)$$
$$N_{26} = l_3^2(\xi)l_2^2(\eta)l_2^2(\zeta) = \frac{1}{2}\xi(\xi+1)(1-\eta^2)(1-\zeta^2)$$
$$N_{27} = l_2^2(\xi)l_2^2(\eta)l_2^2(\zeta) = (1-\xi^2)(1-\eta^2)(1-\zeta^2)$$

Exercise 1 (page 135)

Start with

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Since node 9 is not present, but nodes 5 through 8 are,

$$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Now, adjust N_1 to N_4 so that they vanish at the nodal points.

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

Exercise 2 (page 135)

Coalescing nodes 3, 4, and 7 into node 3, set $\boldsymbol{x}_4^e = \boldsymbol{x}_7^e = \boldsymbol{x}_3^e$.

$$\boldsymbol{x} = \sum_{a=1}^{8} N_a\, \boldsymbol{x}_a^e = N_1\, \boldsymbol{x}_1^e + N_2\, \boldsymbol{x}_2^e + (N_3 + N_4 + N_7)\boldsymbol{x}_3^e + N_5\, \boldsymbol{x}_5^e + N_6\, \boldsymbol{x}_6^e + N_8\, \boldsymbol{x}_8^e$$

So that,

$$N_1{}' = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2{}' = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3{}' = N_3 + N_4 + N_7 = \frac{1}{2}\eta(1+\eta)$$

$$N_5{}' = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6{}' = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_8{}' = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Exercise 3 (page 135)

After adjusting N_1 through N_4, before stopping, check whether any three nodes along an edge is to be collapsed.