ME235A Finite Element Analysis
Fall, 2001

Solution to Problem Set #7
Dec. 7 2001

Exercise 1 (page 142)

	Integration	Gaussian quadrature
1:	$\int_{-1}^{1} 1 d\xi = 2$	$1 \cdot 2 = 2$
x :	$\int_{-1}^{1} \xi d\xi = 0$	$0 \cdot 2 = 0$
x^2 :	$\int_{-1}^{1} \xi^2 d\xi = \frac{2}{3}$	$\left(-\frac{1}{\sqrt{3}}\right)^2 \cdot 1 + \left(\frac{1}{\sqrt{3}}\right)^2 \cdot 1 = \frac{2}{3}$
x^3 :	$\int_{-1}^{1} \xi^3 d\xi = 0$	$\left(-\frac{1}{\sqrt{3}}\right)^3 \cdot 1 + \left(\frac{1}{\sqrt{3}}\right)^3 \cdot 1 = 0$

Exercise 3 (page 145)

l	$l^{(1)}$	$l^{(2)}$	$\tilde{\xi}_l$	$\tilde{\eta}_l$	W_l
1	1	1	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
2	1	2	$-\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$
3	1	3	$-\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
4	2	1	0	$-\sqrt{\frac{3}{5}}$	$\frac{40}{81}$
5	2	2	0	0	$\frac{64}{81}$
6	2	3	0	$\sqrt{\frac{3}{5}}$	$\frac{40}{81}$
7	3	1	$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
8	3	2	$\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$
9	3	3	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{25}{81}$

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Exercise 4 (page 145)

One point

$$\tilde{\xi}_1 = \tilde{\eta}_1 = \tilde{\zeta}_1 = 0\,,\; W_1 = 2\cdot 2\cdot 2 = 8$$

2µ2µ2

l	$\tilde{\xi}_l$	$\tilde{\eta}_l$	$\tilde{\zeta}_l$	W_l
1	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
2	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1
3	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
4	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1
5	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
6	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1
7	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
8	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1

Exercise 1 (page 156)

By taking elements 1 and 2 as 8-node quadrilaterals, and elements 3 and 4 as 4-node quadrilaterals, along the edges of elements 1 and 3, and elements 2 and 4, the condition C2, continuity across element boundaries is violated. This can be corrected by taking elements 3 and 4 to be 5-node quadrilaterals.

Exercise 2 (page 157)

a. f = constant, $x_2^e = \frac{1}{2}\left(x_1^e + x_2^e\right)$

$$x_{,\xi} = \sum_{a=1}^{3} N_{a,\xi} x_a^e = \frac{1}{2}(2\xi - 1)x_1^e - 2\xi x_2^e + \frac{1}{2}(2\xi + 1)x_3^e$$

$$= \frac{\xi}{2}\left(x_1^e - 2x_2^e + x_3^e\right) + \frac{1}{2}\left(x_3^e - x_1^e\right) = \frac{h^e}{2}$$

$$f_a^e = \int_{x_1^e}^{x_3^e} N_a f dx = \frac{fh^e}{2}\int_{-1}^{1} N_a d\xi, \qquad a = 1, 2, 3$$

$$f_1^e = \frac{fh^e}{2}\int_{-1}^{1} \frac{1}{2}\xi(\xi - 1) d\xi = \frac{fh^e}{6}$$

$$f_2^e = \frac{fh^e}{2}\int_{-1}^{1} \left(1 - \xi^2\right) d\xi = \frac{2fh^e}{3}$$

$$f_3^e = \frac{fh^e}{2}\int_{-1}^{1} \frac{1}{2}\xi(\xi + 1) d\xi = \frac{fh^e}{6}$$

b. $f = \delta\left(x - \bar{x}\right),\ x_2^e = \frac{1}{2}\left(x_1^e + x_2^e\right)$

$$f_a^e = \int_{x_1^e}^{x_3^e} N_a f dx = \int_{-1}^{1} N_a \delta\left(\xi - \bar{\xi}\right)\left(x_{,\xi}\right)^{-1} x_{,\xi} d\xi = N_a\left(\bar{\xi}\right), \quad \text{a} = 1, 2, 3$$

where $\bar{x} = x\left(\bar{\xi}\right)$

when $\bar{x} = x_1^e,\ \bar{\xi} = -1$

$$f_1^e = 1,\ f_2^e = f_3^e = 0$$

when $\bar{x} = x_2^e,\ \bar{\xi} = 0$

$$f_1^e = f_3^e = 0,\ f_2^e = 1$$

c. f = constant

$$x_{,\xi} = \frac{\xi}{2}\left(x_1^e - 2x_2^e + x_3^e\right) + \frac{h^e}{2}$$

$$f_a^e = \int_{x_1^e}^{x_3^e} N_a f dx = f\int_{-1}^{1} N_a x_{,\xi} d\xi, \qquad \text{a} = 1, 2, 3$$

$N_a\left(\xi\right)x_{,\xi}\left(\xi\right)$ is a cubic function in x, so in order to integrate exactly using Gaussian quadrature, $n_{int} = 2$.

d. $x_2^e = \frac{1}{2}\left(x_1^e + x_2^e\right)$

$$k_{ab} = \int_{x_1^e}^{x_3^e} N_{a,x} N_{b,x} dx = \int_{-1}^{1} N_{a,\xi} N_{b,\xi}\left(x_{,\xi}\right)^{-1} d\xi = \frac{2}{h^e}\int_{-1}^{1} N_{a,\xi} N_{b,\xi} d\xi$$

$N_{a,\xi} N_{b,\xi}$ is a quadratic function in x, so in order to integrate exactly using Gaussian quadrature, $n_{int} = 2$.